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A Formula for the Grössencharacter of a Parametrized Elliptic Curve

ROBERT S. RUMELY

Department of Mathematics, University of Georgia, Athens, Georgia 30602

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A formula for the grössencharacter of an elliptic curve with complex multiplication, in a family parametrized by modified Weierstrass functions or classical theta-functions, is given. The method is based on Shimura's Reciprocity Law for modular functions, and applies to Legendre, Jacobi, and Hesse curves. As an application, the conductors of the *CM* curves in these families are determined.

The grössencharacter of an elliptic curve with complex multiplication is an important arithmetic invariant. Deuring [2] first proved its existence in showing that the zeta-function of the curve is a product of Hecke L-series. It is ramified precisely where the curve has bad reduction; at a "good" prime its value reduces to the Frobenius endomorphism. Its minimal field of definition is the smallest field over which the torsion points are abelian.

In this paper we give a formula for the grössencharacter of an elliptic curve belonging to an explicitly parametrized family over the moduli space. Our approach is based on Shimura's Reciprocity Law for modular functions, and is different from that of Weil [13] using Jacobi sums. Some families of curves to which our methods apply are those defined by

Legendre's equation	$y^2 = x(x-1)(x-\lambda)$,
Hesse's equation	$x^3 + y^3 + 1 = 3\mu xy$,

and if ϕ is any arithmetic automorphic form of weight 1, the modified Weierstrass equation

$$y^2 = 4x^3 - \frac{g_2}{(2\pi i)^4 \phi^4} x - \frac{g_3}{(2\pi i)^6 \phi^6}.$$

Each of these families has the property that if z_0 is a CM point in the upper half-plane, the curve obtained by evaluating the parameters at z_0 has

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complex multiplication. This is due to the hidden fact that the curves themselves are uniformly parametrized by theta functions or Weierstrass functions. We associate a subgroup of $GL_2(\mathbb{A})$ to such a family, and describe the kernel of the grössencharacter as the pullback of the group under the Reciprocity Law map.

A principle stemming from our work is that the data

Parametrized family of elliptic curves + CM point in moduli space

is sufficient to give the grössencharacter of the corresponding curve in the family. This can be generalized to higher dimensional abelian varieties [5].

The main focus of this paper is on applications. Probably one interest in the formula will be as a source of elliptic curves with known grössencharacter; we provide several numerical examples. A more concrete application is a complete determination of the exponents of the conductors of CM Legendre and Hesse curves.

1. Notation and Preliminaries

The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ mean as usual the ring of rational integers, and the fields of rational, real, and complex numbers, respectively; $\mathfrak{H}$ denotes the complex upper half-plane. All vectors will be column vectors. $\mathbb{1} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

All number fields will be taken to be subfields of $\mathbb{C}$. In general, we use standard notation regarding local and global fields. If K is an algebraic number field, K_{ab} denotes the maximal abelian extension of K, $K_{\mathbb{A}}$ its adele ring, and $K_{\mathbb{A}}^{\times}$ its adele group; we regard K as embedded in $K_{\mathbb{A}}$. $\mathbb{A}$ will be the ring of rational adeles, $\mathbb{A}_f$ its finite part, and $\mathbb{Z}_f = \prod_p \mathbb{Z}_p$ the closure of $\mathbb{Z}$ in $\mathbb{A}_f$. x_∞ will be the archimedean part of an adele x; x_p its component at the prime p. For $s \in K_{\mathbb{A}}^{\times}$, the Artin map acting on K_{ab} will be written $[s, K]$: it is normalized so that its value on local uniformizers is the Frobenius.

We write Γ for $SL_2(\mathbb{Z})$ and $\Gamma(N)$ for its principal congruence subgroup of level N. An automorphic form of weight k ($k \in \mathbb{Z}$) is a function $f(z)$, meromorphic on $\mathfrak{H}$ and on the cusps of $SL_2(\mathbb{Z})$, satisfying $f(z) = (cz+d)^{-k} \cdot f(\gamma(z))$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in some $\Gamma(N)$. It has a Fourier expansion at $i\infty$; if the Fourier coefficients belong to $\mathbb{Q}_{ab}$ it is called arithmetic, and the space of arithmetic automorphic forms of weight k is denoted $\mathfrak{A}_k(\mathbb{Q}_{ab})$.

Let $GL_2(\mathbb{A})_+$ be the set of elements in $GL_2(\mathbb{A})$ whose archimedean component has positive determinant. Shimura [10] has shown that there is a right action of $GL_2(\mathbb{A})_+$ on arithmetic automorphic forms, written $f \to f^u$ for $f \in \mathfrak{A}_k(\mathbb{Q}_{ab})$ and $u \in GL_2(\mathbb{A})_+$, characterized by the properties

(I) $f^\gamma = f|_k \gamma = (cz+d)^{-k} f(\gamma(z))$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(\mathbb{Q})_+$ (regarded as embedded in $GL_2(\mathbb{A})_+$).

(II) $f^{\iota(t)} = f^{[t,\mathbb{Q}]}$ for

$$\iota(t) = \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} \qquad \text{with} \quad t \in \mathbb{Z}_f^{\times},$$

where $f^{[t,\mathbb{Q}]}$ is the automorphic form obtained from f by letting $[t, \mathbb{Q}]$ act on the Fourier coefficients.

(III) The subgroup of $GL_2(\mathbb{A})_+$ fixing f is open.

Note that the factor $\det(\gamma)^{k/2}$ one frequently sees in the definition of $f|_k \gamma$ is omitted here; this is crucial for our theory, as it means forms of weight $k \geqslant 1$ are not fixed by $\mathbb{Q}^{\times} \cdot \mathbb{1}$. An important subgroup of $GL_2(\mathbb{A})_+$ is

$$\mathbb{U} = GL_2(\mathbb{R})_+ \times \prod_p GL_2(\mathbb{Z}_p).$$

It is the largest subgroup with compact finite part fixing the modular invariant $j(z)$.

Elements of $\mathcal{O}\!\mathcal{l}_0(\mathbb{Q}_{ab})$ are called arithmetic automorphic functions. If $z_0 \in \mathfrak{H}$ is a CM point—that is, z_0 belongs to a quadratic imaginary field K—and $f(z) \in \mathcal{O}\!\mathcal{l}_0(\mathbb{Q}_{ab})$ is finite at z_0, then $f(z_0)$ lies in K_{ab}. Shimura's Reciprocity Law gives the action of $K_{\mathbb{A}}^{\times}$ on $f(z_0)$ in terms of the action of $GL_2(\mathbb{A})_+$ on $\mathcal{O}\!\mathcal{l}_0(\mathbb{Q}_{ab})$, as follows. There is a natural embedding $q_{z_0}: K \to M_2(\mathbb{Q})$ representing multiplication in K with respect to the basis $z_0, 1$:

$$q_{z_0}(\kappa) \begin{pmatrix} z_0 \\ 1 \end{pmatrix} = \kappa \begin{pmatrix} z_0 \\ 1 \end{pmatrix} \qquad \text{for} \quad \kappa \in K.$$

Extending scalars, one obtains the Reciprocity Law map $q_{z_0}: K_{\mathbb{A}}^{\times} \to GL_2(\mathbb{A})_+$. Shimura's Reciprocity Law is the formula

$$f(z_0)^{[s,K]} = f^{q_{z_0}(s^{-1})}(z_0) \qquad \text{for} \quad s \in K_{\mathbb{A}}^{\times}$$

(cf. [8, Theorem 6.31, p. 157]).

2. The Main Theorems

Let ϕ be an arithmetic automorphic form of weight 1; put

$$\gamma_2(z) = \frac{g_2(z)}{(2\pi i)^4\, \phi^4(z)}, \qquad \gamma_3(z) = \frac{g_3(z)}{(2\pi i)^6\, \phi^6(z)},$$

and consider the family of elliptic curves

$$W_\phi : y^2 = 4x^3 - \gamma_2 x - \gamma_3$$

as z varies over $\mathfrak{H}$. For each point z_0 in a quadratic imaginary field K, where $\gamma_2(z_0)$ and $\gamma_3(z_0)$ are finite, the corresponding curve $W_\phi(z_0)$ has complex multiplication by K.

The family W_ϕ is parametrized by modified Weierstrass functions: for each $z \in \mathfrak{H}$, as w varies over $\mathbb{C}$ there is a holomorphic isomorphism $\mathscr{W}_\phi(w, z): \mathbb{C}/[z, 1] \to W_\phi(z)$,

$$\mathscr{W}_\phi(w, z) = \left(\frac{\wp(w, [z, 1])}{(2\pi i)^2 \phi^2(z)}, \frac{\wp'(w, [z, 1])}{(2\pi i)^3 \phi^3(z)} \right).$$

On the other hand, for fixed $r, s \in \mathbb{Q}/\mathbb{Z}$ not both zero, as z varies over $\mathfrak{H}$,

$$z \to \mathscr{W}_\phi(zr + s, z)$$

is a section whose coordinates are arithmetic automorphic functions. Thus at a CM point z_0, the torsion points of $W_\phi(z_0)$ are rational over K_{ab}. Moreover, writing $\mathbf{v} = {}^t(r, s)$, the functions

$$p(\mathbf{v}, z) = \frac{1}{(2\pi i)^2} \wp(zr + s, [z, 1]), \qquad p'(\mathbf{v}, z) = \frac{1}{(2\pi i)^3} \wp'(zr + s, [z, 1])$$

are arithmetic modular forms on which the subgroup $\mathbb{U}$ of $GL_2(\mathbb{A})_+$ acts in a simple linear way: for $u \in \mathbb{U}$ and $\mathbf{v} \in \mathbb{Q}^2/\mathbb{Z}^2$, ${}^t u\mathbf{v}$ is well defined (cf. [8, p. 145]), and

$$p(\mathbf{v}, z)^u = p({}^t u\mathbf{v}, z), \qquad p'(\mathbf{v}, z)^u = p'({}^t u\mathbf{v}, z).$$

This can be seen by examining the effect of $SL_2(\mathbb{Z})$ on the series defining $\wp(w, [z, 1])$ and $\wp'(w, [z, 1])$, and of $\iota(\mathbb{Z}_f^\times)$ on the q-expansions. For the $p(\mathbf{v}, z)$ it is equivalent to [8, Proposition 6.21, p. 147]. (Note that in [8] vectors are row vectors.)

THEOREM 1. *Let $\mathbb{U}_\phi$ be the subgroup of $\mathbb{U}$ fixing ϕ, let $z_0 \in \mathfrak{H} \cap K$ be a CM point where $\gamma_2(z_0)$ and $\gamma_3(z_0)$ are defined, and consider $W_\phi(z_0)$ over the field of definition $k = K(\gamma_2(z_0), \gamma_3(z_0))$. Then*

(1) *k is classfield to the subgroup $K^\times \cdot q_{z_0}^{-1}(\mathbb{U}_\phi)$ of $K_{\mathbb{A}}^\times$.*

(2) *For each $s \in k_{\mathbb{A}}^\times$, there is a unique decomposition $N_{k/K}(s) = \kappa \cdot \mu$, with $\kappa = \kappa(s) \in K^\times$, $\mu = \mu(s) \in q_{z_0}^{-1}(\mathbb{U}_\phi)$.*

(3) *The grössencharacter of* $W_\phi(z_0)$ *is given by*

$$\chi(s) = \left(\frac{\kappa(s)}{N_{\mathfrak{k}/K}(s)}\right)_\infty \qquad \textit{for} \quad s \in \mathfrak{k}_{\mathbb{A}}^\times .$$

Proof. The fact that $\mathfrak{k} = K(\gamma_2(z_0), \gamma_3(z_0))$ is classfield to $K^\times \cdot q_{z_0}^{-1}(\mathbb{U}_\phi)$ follows from Shimura's Reciprocity Law once it is known that $\mathbb{Q}^\times \cdot \mathbb{U}_\phi$ is the subgroup of $GL_2(\mathbb{A})_+$ fixing $\mathbb{Q}(\gamma_2, \gamma_3)$ (see [8, 6.7.5, 6.23, 6.33]). If $u \in GL_2(\mathbb{A})_+$ fixes γ_2 and γ_3, it fixes $j(z)$. Hence it lies in $\mathbb{Q}^\times \cdot \mathbb{U}$ and may be assumed to be in $\mathbb{U}$. It also fixes $\gamma_2/\gamma_3 = ((2\pi i)^2 g_2/g_3)\phi^2$, so being in $\mathbb{U}$ fixes ϕ^2. Thus $\phi^u = \pm\phi$. Since ϕ is of odd weight, $\phi^{-1} = -\phi$, which gives $\pm u \in \mathbb{U}_\phi$. Conversely, since $\mathbb{U}$ fixes $g_2/(2\pi i)^4$ and $g_3/(2\pi i)^6$, $\mathbb{Q}^\times \cdot \mathbb{U}_\phi$ fixes γ_2 and γ_3.

Thus, for any $s \in \mathfrak{k}_{\mathbb{A}}^\times$ there are $\kappa \in K^\times$ and $\mu \in q_{z_0}^{-1}(\mathbb{U}_\phi)$ such that $N_{\mathfrak{k}/K}(s) = \kappa \cdot \mu$. On the other hand, by the Fundamental Theorem of Complex Multiplication (see [8, Proposition 7.40, p. 211]) there is some $\kappa' = \kappa'(s) \in K^\times$ for which $\chi(s) = (\kappa'(s)/N_{\mathfrak{k}/K}(s))_\infty$; it is unique and is defined by the property that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{Q}^2/\mathbb{Z}^2 \arrow[r, "{[z_0, 1]}"] \arrow[d, "{{}^t q_{z_0}(\kappa' \cdot N_{\mathfrak{k}/K}(s)^{-1})}"'] & K/[z_0, 1] \arrow[r, "\mathscr{W}_\phi"] \arrow[d] & W_\phi(z_0) \arrow[d, "{[s, \mathfrak{k}],}"] \\
\mathbb{Q}^2/\mathbb{Z}^2 \arrow[r, "{[z_0, 1]}"] & K/[z_0, 1] \arrow[r, "\mathscr{W}_\phi"] & W_\phi(z_0)
\end{tikzcd} \qquad s \in \mathfrak{k}_{\mathbb{A}}^\times$$

Since $[s, \mathfrak{k}] \mid K_{ab} = [\mu, K]$ and $q_{z_0}(\mu)$ fixes ϕ, the diagram and Shimura's Reciprocity Law give

$$\begin{aligned}
&\mathscr{W}_\phi({}^t q_{z_0}(\kappa' \cdot \kappa^{-1}\mu^{-1})\,\mathbf{v}, z_0) \\
&= \mathscr{W}_\phi(\mathbf{v}, z_0)^{[\mu, K]} \\
&= \left(\left(\frac{p(\mathbf{v}, z)}{\phi^2(z)}\right)^{q_{z_0}(\mu^{-1})}, \left(\frac{p'(\mathbf{v}, z)}{\phi^3(z)}\right)^{q_{z_0}(\mu^{-1})} \right) \Bigg|_{z_0} \\
&= \mathscr{W}_\phi({}^t q_{z_0}(\mu^{-1})\,\mathbf{v}, z_0)
\end{aligned}$$

for all $\mathbf{v} \in \mathbb{Q}^2/\mathbb{Z}^2$. Hence $\kappa = \kappa'$. ∎

The critical ingredient in the proof was the action of $\mathbb{U}$ on Weierstrass functions. Shimura [10] has recently provided an action of (a covering of) $GL_2(\mathbb{A})_+$ on classical theta-functions. Using it, a similar theorem can be proved for families of curves parametrized by theta-functions; in particular, this applies to the Legendre, Jacobi, and Hesse curves.

For fixed $r, s \in \mathbb{Q}$, the classical theta-function $\theta(u, z; r, s)$ with $u \in \mathbb{C}$, $z \in \mathfrak{H}$, is defined by

$$\theta(u, z; r, s) = \sum_{m \in \mathbb{Z}} e(2\pi i(\tfrac{1}{2}(m + r)^2 z + (m + r)(u + s))).$$

We will say that a family of elliptic curves is parametrized by classical theta-functions if there are modular functions $f_i(z) \in \mathcal{O}_0(\mathbb{Q}_{ab})$ and theta-functions $\theta(u, z; r_{i,j}, s_{i,j})$, $1 \leqslant j \leqslant k$, $0 \leqslant i \leqslant m$, such that for all z in an open subset of $\mathfrak{H}$, the map $\mathscr{F}_z(u)\colon \mathbb{C} \to \mathbb{P}^m$ given by

$$\mathscr{F}_z(u) = \mathscr{F}(u, z) = \left(f_i(z) \prod_{j=1}^{k} \theta(u, z; r_{i,j}, s_{i,j})\right)_{0 \leqslant i \leqslant m},$$

is a nonsingular projective embedding of an elliptic curve $F(z)$. The following theorem is proved in [5]; for now we simply state it and proceed to applications.

THEOREM 2. *Let F be a family of elliptic curves parametrized by classical theta-functions. The curve $F(z)$ corresponding to $z \in \mathfrak{H}$ has complex multiplication iff $z = z_0$ lies in a quadratic imaginary field K. There exists a subgroup $\mathbb{U}_F$ of $GL_2(\mathbb{A})_+$ such that for every CM point z_0, where $F(z_0)$ is defined,*

(1) *The classfield k to $K^\times \cdot q_{z_0}^{-1}(\mathbb{U}_F)$ is a field of definition for $F(z_0)$.*

(2) *For each $s \in k_{\mathbb{A}}^\times$, there is a unique decomposition $N_{k/K}(s) = \kappa \cdot \mu$ with $\kappa \in K^\times$, $\mu \in q_{z_0}^{-1}(\mathbb{U}_F)$.*

(3) *The grössencharacter of $F(z_0)/k$ is given by*

$$\chi(s) = \left(\frac{\kappa(s)}{N_{k/K}(s)}\right)_\infty \qquad \textit{for} \quad s \in k_{\mathbb{A}}^\times.$$

EXAMPLE 1. Take $\phi = \eta^2$, where $\eta = e(\pi i z/12) \prod_{n=1}^{\infty}(1 - e(2\pi i n z))$ is the Dedekind eta function, and consider the Weierstrass family W_{η^2} defined by

$$y^2 = 4x^3 - \frac{g_2}{(2\pi i)^4 \eta^8} x - \frac{g_3}{(2\pi i)^6 \eta^{12}}.$$

Since η^2 has rational Fourier coefficients and is fixed by precisely the commutator subgroup Γ_c of $SL_2(\mathbb{Z})$, which contains $\Gamma(12)$, by the Strong Approximation Theorem

$$\mathbb{U}_{\eta^2} = \iota(\mathbb{Z}_f^\times) \cdot \Gamma_c \cdot \mathbb{U}(12),$$

where we have written $\mathbb{U}(12)$ for $\{u \in \mathbb{U} \mid u_p \equiv \mathbb{1} \bmod 12M_2(\mathbb{Z}_p) \text{ for all } p\}$. $\mathbb{U}_{\eta^2}$ has index 12 in $\mathbb{U}$, the minimal possible index for a family defined at every point of $\mathfrak{H}$.

The following examples are taken from [5].

EXAMPLE 2. For the Legendre curves $y^2 = x(x-1)(x-\lambda)$ and the Jacobi curves $y^2 = (1-x^2)(1-k^2x^2)$ the group is

$$\mathbb{U}_{LJ} = \left\{ u \in \mathbb{U} \;\middle|\; u_2 = \begin{pmatrix} a_2 b_2 \\ c_2 d_2 \end{pmatrix} \equiv \mathbb{1} \bmod 2M_2(\mathbb{Z}_2), d_2 \equiv 1 \bmod 4\mathbb{Z}_2 \right\}$$

and the field of definition $\mathscr{k}$ in the theorem is precisely $K(\lambda(z_0))$. Here $\lambda = \lambda(z) = k^2(z)$ is the modular function of level 2 satisfying $j = 2^8(\lambda^2 - \lambda + 1)^3/\lambda^2(\lambda-1)^2$ and taking values $(1, \infty, 0)$ at the cusps $(0, 1, \infty)$.

EXAMPLE 3. For the Hesse curves $x^3 + y^3 + 1 = 3\mu xy$ the group is

$$\mathbb{U}_H = \left\{ u \in \mathbb{U} \;\middle|\; u_3 = \begin{pmatrix} a_3 b_3 \\ c_3 d_3 \end{pmatrix} \equiv \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3M_2(\mathbb{Z}_3) \right\}$$

and $\mathscr{k} = K(\mu(z_0))$; $\mu(z)$ is the modular function of level 3 satisfying $j = 3^3\mu^3(\mu^3+2)^3/(\mu^3-1)^3$ with values $(e^{-2\pi i/3}, \infty, e^{2\pi i/3}, 1)$ at the cusps $(-1, 0, 1, \infty)$.

3. Arithmetic Information About Specific Curves

The basis of our applications is that $q_{z_0}^{-1}(\mathbb{U}_F)$ is essentially the kernel of the grössencharacter. It can be computed using the following semi-local formulas for the embeddings $q_{z_0}\colon (\mathbb{Q}_p \otimes K)^\times \to GL_2(\mathbb{Q}_p)$. Take $0 < d \in \mathbb{Z}$ square-free, write $z_0 = \ell + m\sqrt{d}i$, with $\ell, m \in \mathbb{Q}$, $m > 0$; put $K = \mathbb{Q}(z_0)$, and take $a, b \in \mathbb{Q}_p$. At a prime which is ramified or inert in K, $\mathbb{Q}_p(\sqrt{-d}) = \mathbb{Q}_p \otimes K$. Identifying $\sqrt{-d}$ with $1 \otimes \sqrt{d}i$, one has

$$q_{z_0}(a + b\sqrt{-d}) = \begin{bmatrix} a + \dfrac{b}{m}\ell & -\dfrac{b}{m}(\ell^2 + m^2 d) \\ \dfrac{b}{m} & a - \dfrac{b}{m}\ell \end{bmatrix}. \tag{1}$$

For a prime which splits in K, let $\sqrt{-d}$ denote a fixed square root of $-d$ in $\mathbb{Q}_p$. $\mathbb{Q}_p \otimes K$ contains two idempotents α_1 and α_2. If $\delta \in \mathbb{Z}$ is congruent to $\sqrt{-d} \bmod 2p\mathbb{Z}_p$, and α_1 corresponds to the prime $\mathfrak{p}$ for which $\operatorname{ord}_{\mathfrak{p}}((\delta - \sqrt{d}i)/2) > 0$, then for $a, b \in \mathbb{Q}_p$

$$q_{z_0}(a\alpha_1 + b\alpha_2) = \begin{bmatrix} \frac{a+b}{2} - \frac{a-b}{2md}\ell\sqrt{-d} & \frac{a-b}{2md}\sqrt{-d}(\ell^2 + m^2 d) \\ -\frac{a-b}{2md}\sqrt{-d} & \frac{a+b}{2} + \frac{a-b}{2md}\ell\sqrt{-d} \end{bmatrix}. \quad (2)$$

These formulas may be verified using the definition of q_{z_0} and a computation of the idempotents; $\alpha_1 = \frac{1}{2}(1 \otimes 1 - (1/d)(\sqrt{-d} \otimes \sqrt{d}i))$.

In many cases the special values of modular functions are known from tables in Weber [12]. From them we obtain examples of curves with complex multiplication by a particular order and with known grössencharacter χ. By Deuring's result [2], the zeta function of the curve over a field of definition k containing the CM field is $L(s, \chi) \cdot L(s, \bar{\chi})$. If the curve is defined over $\mathbb{Q}$ or K, $L(s, \chi)$ corresponds to a modular form $f(z)$, and $f(z)$ can be determined. A lemma of Shimura, [9, Lemma 3, p. 203], making precise earlier work of Hecke, is useful in this regard. Let $-D$ be the discriminant of K, and put $M = D \cdot N(c)$, where $N(c)$ is the norm of the conductor of χ. Let ε be the character of $(\mathbb{Z}/M\mathbb{Z})^\times$ defined by

$$\varepsilon(d) = \left(\frac{-D}{d}\right) \frac{\chi((d))}{d} \qquad \text{for} \quad d \in \mathbb{Z}, (d, M) = 1.$$

Then $f(z)$ is an element of $S_2(M, \varepsilon)$, the space of cusp forms $g(z)$ of weight 2 satisfying $g\,|_2\, \gamma = \varepsilon(d)\, g$ for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & b \end{smallmatrix}\right) \in \Gamma_0(M)$.

The examples below were obtained by analyzing $q_{z_0}^{-1}(\mathbb{U}_F)$; the method is illustrated only for the first one. The conductors are those of the grössencharacter χ; the kernel of χ in $K_{\mathbb{A}}^\times$ and the modular form are given also, when the field of definition k coincides with the CM field K. (a/b) denotes the Jacobi symbol.

(a) The Legendre curve for $z_0 = i$: $y^2 = x(x-1)(x-\frac{1}{2})$. $k = K = \mathbb{Q}(i)$; complex multiplication by the maximal order of K; $j = 2^6 \cdot 3^3$. Conductor $= \mathfrak{p}_2^4$. $\operatorname{Ker}(\chi) = \mathbb{C}^\times \times \{\{1, 1+2i\} \cdot (1 + \mathfrak{p}_2^4)\} \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_{\mathfrak{p}}$ (here we are writing $\mathscr{U}_{\mathfrak{p}}$ for the group of local units at $\mathfrak{p}$). The modular form is number 64B in the Antwerp IV tables [1, p. 117]; it has level $M = 64$ and character $\varepsilon(d) = 1$.

Weber [12, p. 179] gives $k^2(z) = (f_2(z)/f(z))^8$, where [12, pp. 721; 426] $f_2(i) = \sqrt[8]{2}$, $f(i) = \sqrt[4]{2}$; hence $k^2(i) = \frac{1}{2}$. Since the field of definition k is $\mathbb{Q}(i) = K$, $\ker(\chi) = q_{z_0}^{-1}(\mathbb{U}_{LJ})$, and using the semi-local formulas for q_{z_0} and the description of $\mathbb{U}_{LJ}$, one computes the components

$$q_{z_0}^{-1}(\mathbb{U}_{LJ})_2 = \{a + bi \mid a, b \in \mathbb{Z}_2, a \equiv 1 \bmod 4, b \equiv 0 \bmod 2\},$$

$$q_{z_0}^{-1}(\mathbb{U}_{LJ})_p = \{\text{all units}\} \text{ for other primes } p.$$

Expressed in coordinate-free notation this is

$$\mathrm{Ker}(\chi) = \mathbb{C}^\times \times \{\{1, 1+2i\} \cdot (1 + \mathfrak{p}_2^4)\} \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_\mathfrak{p}.$$

Clearly the conductor of χ is $\mathfrak{p}_2^4$. To evaluate χ, regarded as a Hecke character on ideals, note that $\chi(\mathfrak{a})$ is that choice of a generator for $\mathfrak{a}$ which lies in $q_{z_0}^{-1}(\mathbb{U}_{LJ})_2$. Thus, for odd rational integers d, $\chi((d)) = (-1)^{(d-1)/2} d = (-1/d)d$. By Shimura's lemma, the modular form associated to the curve has level $M = 64$ and character $\varepsilon(d) = 1$.

(b) The Legendre curve for $z_0 = \sqrt{5}i$: $y^2 = x(x-1)(x-\lambda)$, where $\lambda = \frac{1}{2} - \frac{1}{4}(3 - \sqrt{5})\sqrt{2 + 2\sqrt{5}}$ (cf. [12, p. 721]). $k = K(\lambda)$ is a cyclic extension of $K = \mathbb{Q}(\sqrt{5}i)$ of degree 4; $[K(\lambda) : K(j)] = 2$, $j = 2^6 \cdot 5 \cdot (1975 + 884\sqrt{5})$. Complex multiplication by the maximal order of $\mathbb{Q}(\sqrt{5}i)$; by Section 4, Table I, the conductor is $\mathfrak{P}_2^6$, where the prime $\mathfrak{P}_2$ of k lies over the rational prime (2).

(c) The Hesse curve for $z_0 = (3 + \sqrt{3}i)/2$: $x^3 + y^3 + 1 = 0$. This is the Fermat curve, which Weil treated using Jacobi Sums. $k = K = \mathbb{Q}(\zeta_3)$; $j = 0$, *CM* by the maximal order. Conductor $= \mathfrak{p}_3^2$; $\mathrm{Ker}(\chi) = \mathbb{C}^\times \times (1 + \mathfrak{p}_3^2) \times \prod_{\mathfrak{p} \nmid 3} \mathscr{U}_\mathfrak{p}$. The modular form is 27B in the Antwerp IV tables: $M = 27$, $\varepsilon(d) = 1$.

In the following two examples the value of $\mu(z_0)$ was obtained from the known value of $j(z)$ by factoring the polynomial resulting from $j = 27\mu^3(\mu^3 + 2)^3/(\mu^3 - 1)^3$, and using geometrical considerations to determine the root corresponding to z_0.

(d) The Hesse curve for $z_0 = (4 + \sqrt{2}i)/3$: $x^3 + y^3 + 1 = (-2 + \sqrt{2}i)\, xy$. $j = 2^6 \cdot 5^3$; $k = K = \mathbb{Q}(\sqrt{2}i)$; *CM* by the maximal order. By Section 4, Table II, the conductor is $\mathfrak{p}_{3,1}$ and $\mathrm{Ker}(\chi) = \mathbb{C}^\times \times (1 + \mathfrak{p}_{3,1}) \times \mathscr{U}_{\mathfrak{p}_{3,2}} \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_\mathfrak{p}$, where $\mathfrak{p}_{3,1} = (1 + \sqrt{2}i)$ and $\mathfrak{p}_{3,2} = (1 - \sqrt{2}i)$. The modular form has level $M = 24$, character $\varepsilon(d) = (24/d)$.

(e) The Hesse curve for $z_0 = (5 + \sqrt{11}i)/6$: $x^3 + y^3 + 1 = (-1 + \sqrt{11}i)\, xy$. $j = -2^{15}$; $k = K = \mathbb{Q}(\sqrt{11}i)$; *CM* by the maximal order. Conductor $= \mathfrak{p}_{3,1}$, $\mathrm{Ker}(\chi) = \mathbb{C}^\times \times (1 + \mathfrak{p}_{3,1}) \times \mathscr{U}_{\mathfrak{p}_{3,2}} \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_\mathfrak{p}$, where $\mathfrak{p}_{3,1} = ((1 - \sqrt{11}i)/2)$, $\mathfrak{p}_{3,2} = ((1 + \sqrt{11}i)/2)$. The modular form has level $M = 33$; $\varepsilon(d) = (33/d)$.

For the Weierstrass family with $\phi = \eta^2$, one has $j(z) = (12\gamma_2(z))^3 = (216\gamma_3(z))^2 + 12^3$, and these relations plus geometry were used to find $\gamma_2(z_0)$ and $\gamma_3(z_0)$ for the curves below.

(f) The η^2-Weierstrass curve for $z_0 = (3 + \sqrt{7}i)/2$: $y^2 = 4x^3 + \frac{5}{4}x + (\sqrt{7}i/8)$. $k = K = \mathbb{Q}(\sqrt{7}i)$; $j = -3^3 \cdot 5^3$. *CM* by the maximal order. $\mathrm{Ker}(\chi) = \mathbb{C}^\times \times \mathscr{U}_{\mathfrak{p}_{2,1}} \times (1 + \mathfrak{p}_{2,2}^2) \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_\mathfrak{p}$, where $\mathfrak{p}_{2,1} = ((1 + \sqrt{7}i)/2)$, $\mathfrak{p}_{2,2} = ((1 - \sqrt{7}i)/2)$. Conductor $= \mathfrak{p}_{2,2}^2$. The modular form is of level $M = 28$ with character $\varepsilon(d) = (28/d)$.

(g) The η^2-Weierstrass curve for $z_0 = \sqrt{7}i$: $y^2 = 4x^3 - \frac{85}{4}x + \frac{57}{8}\sqrt{7}$. $\mathscr{k} = K(\sqrt{7})$; $[\mathscr{k}: K(j)] = 2$. *CM* by the order of $\mathbb{Q}(\sqrt{7}i)$ with conductor 2; $j = 3^3 \cdot 5^3 \cdot 17^3$. The curve has good reduction at all primes of $\mathbb{Q}(\sqrt{7})$: the conductor is (1).

(h) The η^2-Weierstrass curve for $z_0 = (3 + \sqrt{163}i)/2$: $y^2 = 4x^3 + 80 \cdot 23 \cdot 29x + 7 \cdot 11 \cdot 19 \cdot 127\sqrt{163}i$. $\mathscr{k} = K = \mathbb{Q}(\sqrt{163}i)$; *CM* by the maximal order; $j = -2^{18} \cdot 3^3 \cdot 5^3 \cdot 23^3 \cdot 29^3$. Conductor $= \mathfrak{p}_2^2$, where $\mathfrak{p}_2 = (2)$. $\operatorname{Ker}(\chi) = \mathbb{C}^\times \times (\{(3 + \sqrt{163}i)/2\} \cdot (1 + \mathfrak{p}_2^2)) \times \prod_{\mathfrak{p} \nmid 2} \mathscr{U}_\mathfrak{p}$; the factor in brackets means the subgroup of $\mathscr{U}_{\mathfrak{p}_2}$ generated over $1 + \mathfrak{p}_2^2$ by $(3 + \sqrt{163}i)/2$. The modular form has level $M = 16 \cdot 163$, and $\varepsilon(d) = (163/d)$.

(i) The η^2-Weierstrass curve for $z_0 = \sqrt{3}i$: $y^2 = 4x^3 - \frac{5}{2}\sqrt[3]{2}x + \frac{11}{13}\sqrt{3}$. Here $K = \mathbb{Q}(\sqrt{3}i)$, $\mathscr{k} = K(\sqrt[3]{2}, \sqrt{3})$ and $[\mathscr{k}: K] = 6$. Complex multiplication by the order of K with conductor 2; $j = 2^4 \cdot 3^3 \cdot 5^3$. The grössencharacter factors through $q_{z_0}^{-1}(\mathbb{U}_{\eta^2}) = \mathbb{C}^\times \times [\{1 + 2\sqrt{3}i\} \cdot (1 + \mathfrak{p}_2^3)] \times [\{\pm 1\} \cdot (1 + \mathfrak{p}_3^2)] \times \prod_{\mathfrak{p} \nmid 6} \mathscr{U}_\mathfrak{p}$; by the methods of Section 4, the conductor of the grössencharacter is $\mathfrak{P}_2^4$, where $\mathfrak{P}_2$ is the unique prime of $\mathscr{k}$ lying over (2).

4. The Conductors of the Legendre and Hesse Curves

As an example of more general information that can be obtained from the formula, in this section we will compute the exponents of the conductors of the grössencharacters of Legendre and Hesse curves. The main difficulty is that the grössencharacter factors as

$$\chi: \mathscr{k}_\mathbb{A}^\times \xrightarrow{N_{\mathscr{k}/K}} K_\mathbb{A}^\times \xrightarrow{\phi} \mathbb{C}^\times$$

and while we have a good description of ϕ, the field $\mathscr{k}$ is only given by class-field theory. However, higher ramification theory provides the information about the norm map $N_{\mathscr{k}/K}$ that we need. Recall that if L is a local field with prime $\mathfrak{P}$ and group of units $\mathscr{U}_L$, then for a real number v, if m is the least integer $\geqslant v$, $\mathscr{U}_L^v = \mathscr{U}_L \cap (1 + \mathfrak{P}^m)$.

Lemma. *Let L/M be an abelian extension of local fields, and $\psi = \psi_{L/M}$ the Herbrand function for this extension. Then, if m is an integer and $\psi(m-1) < v \leqslant \psi(m)$ one has*

$$N_{L/M}(\mathscr{U}_L^v) = \mathscr{U}_M^m \cap N_{L/M}(L^\times).$$

Proof. By Serre [6, Theorem 2, p. 235 and Theorem 1.c, p. 234], $N_{L/M}(\mathscr{U}_L^{\psi(m)}) = \mathscr{U}_M^m \cap N_{L/M}(L^\times)$. On the other hand, by Proposition 2, p. 90

and Proposition 8, p. 99 of the same source, $N_{L/M}(\mathscr{U}_L^{\psi(m-1)+1}) \subset \mathscr{U}_M^m$. Since ψ is monotone increasing and takes integers to integers, we get the result. ∎

Define $\psi_{\mathfrak{P}}$ to be the exponent of the conductor of the grössencharacter at a prime $\mathfrak{P}$ of k: that is, $\psi_{\mathfrak{P}}$ is the least integer k such that χ is trivial on $\mathscr{U}_{\mathfrak{P}}^k$ (regarded as embedded in $k_{\mathbb{A}}^{\times}$). $\psi_{\mathfrak{P}}$ carries information about the type of reduction of the curve. (See Tate [11]. Note that the exponent of the conductor of the curve, as used in Tate [11], and Ogg [4], is twice the exponent of the conductor of the grössencharacter, cf. [7, Theorem 12, p. 514].) Suppose $\mathfrak{p}$ is the prime of K below $\mathfrak{P}$; regard $K^{\times}$ as embedded in $K_{\mathbb{A}}^{\times}$, and put $H_{z_0} = q_{z_0}^{-1}(\mathbb{U}_F)$. By class field theory

$$N_{k_{\mathfrak{P}}/K_{\mathfrak{p}}}(k_{\mathfrak{P}}^{\times}) = K_{\mathfrak{p}}^{\times} \cap N_{k/K}(k_{\mathbb{A}}^{\times}) = K_{\mathfrak{p}}^{\times} \cap K^{\times} \cdot H_{z_0}.$$

Since H_{z_0} is the kernel of χ, the lemma tells us that $\psi_{\mathfrak{P}} = \psi_{k_{\mathfrak{P}}/K_{\mathfrak{p}}}(m-1)+1$, where m is the least integer such that

$$\mathscr{U}_{\mathfrak{p}}^m \cap K^{\times} \cdot H_{z_0} = \mathscr{U}_{\mathfrak{p}}^m \cap H_{z_0}.$$

To find $\psi_{\mathfrak{P}}$, then, compute $H_{z_0} = q_{z_0}^{-1}(\mathbb{U}_F)$ using formulas (1) and (2); compute the "norm group" $V_{\mathfrak{p}} = \mathscr{U}_{\mathfrak{p}} \cap K^{\times} \cdot H_{z_0}$ and the "kernel group" $W_{\mathfrak{p}} = \mathscr{U}_{\mathfrak{p}} \cap H_{z_0}$; compute the Herbrand function, implicit in Serre [6, Proposition 13, p. 81] as

$$\psi_{k_{\mathfrak{P}}/K_{\mathfrak{p}}}(q) = \sum_{j=1}^{q} [\mathscr{U}_{\mathfrak{p}} : \mathscr{U}_{\mathfrak{p}}^j V_{\mathfrak{p}}] \qquad \text{for} \quad 0 \leqslant q \in \mathbb{Z};$$

and find the least m such that $\mathscr{U}_{\mathfrak{p}}^m \cap V_{\mathfrak{p}} = \mathscr{U}_{\mathfrak{p}}^m \cap W_{\mathfrak{p}}$. In our situation $\psi_{\mathfrak{P}}$ depends only on $\mathfrak{p}$, so we will write $\psi_{\mathfrak{p}}$ for it:

$$\psi_{\mathfrak{p}} = \psi_{k_{\mathfrak{P}}/K_{\mathfrak{p}}}(m-1)+1.$$

We have carried out this procedure for the Legendre and Hesse curves, expressing $\psi_{\mathfrak{p}}$ as a function of the CM point z_0. The computations are very lengthy; only the results can be given here. For the Legendre curves the grössencharacter is ramified only at primes above 2; for the Hesse curves, only at primes above 3. We write $z_0 = \ell + m\sqrt{-d}$, where $\ell, m \in \mathbb{Q}$, $0 < m$, and assume that d is positive, integral, and square-free. Also put $n = N_{K/\mathbb{Q}}(z_0) = \ell^2 + m^2 d$. If t is a rational number and p is a prime, we denote $t_p = \operatorname{ord}_p(t)$. $\psi_{\mathfrak{p}}$ of course depends on congruence conditions on ℓ, m, and d; but it is usefully classified through the conductor c of the order corresponding to the lattice $[z_0, 1]$. One can determine c from z_0 in either of the following ways:

(A) Using the formulas for ord values

$$\begin{aligned} c_2 &= \max(m_2, m_2 - n_2) && \text{if} \quad d \equiv 1, 2 \bmod 4, \\ &= \max(m_2 + 1, m_2 - \ell_2, m_2 - n_2 + 1) && \text{if} \quad d \equiv 3 \bmod 4, \\ c_p &= \max(m_p, m_p - \ell_p, m_p - n_p) && \text{for} \quad p \geqslant 3. \end{aligned}$$

By definition, $q_{z_0}\colon K \to M_2(\mathbb{Q})$ represents multiplication with respect to the basis z_0, 1. Hence, $\mathcal{O} = q_{z_0}^{-1}(M_2(\mathbb{Z}))$, and the formulas arise by computing $q_{z_0}^{-1}(M_2(\mathbb{Z}_p))$ using Eq. (1).

(B) If z_0 satisfies the irreducible equation

$$Ax^2 + Bx + C = 0 \qquad \text{with} \quad (A, B, C) = 1,$$

then $c^2 D = B^2 - 4AC$, where D is the discriminant of $K = \mathbb{Q}(z_0)$. This formula is implicit in Lang [3, Theorem 1, p. 90].

We remark that in Theorem 3, the condition distinguishing the "high" and "low" conductor classes is simple enough, but in our proof it arises by synthesis from a large number of cases. It would be desirable to see what is really behind this.

In both Theorems 3 and 4, we must make a special case when c is a power of a prime and $K = \mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$. The necessity of this was noted by Serre–Tate [7, Theorem 9, p. 507]; here it arises because the extra global units contribute to a larger local norm group when the grössencharacter is ramified at only one prime of K.

THEOREM 3. *For the Legendre curve* $y^2 = x(x-1)(x - \lambda(z_0))$, *the exponent of the conductor of the grössencharacter at a prime* $\mathfrak{P}$ *of* $k = K(\lambda(z_0))$, $\mathfrak{P} \mid \mathfrak{p}$ *of* K, $\mathfrak{p} \mid 2$, *is given by* $\psi_\mathfrak{p}$ *in Table* I; *for each field* $K = \mathbb{Q}(\sqrt{-d})$, *the curves divide into a "high conductor" class where* $\psi_\mathfrak{p}$ *varies with the conductor* c *of the order corresponding to* $[z_0, 1]$, *and a "low conductor" class where it does not. The high conductor class is distinguished by the condition that* $\operatorname{ord}_\mathfrak{p}(z_0) = 0$ *for every prime* $\mathfrak{p}$ *of* K *lying over* (2).

THEOREM 4. *For the Hesse curve* $x^3 + y^3 + 1 = 3\mu(z_0)\, xy$, *the exponent of the conductor of the grössencharacter at a prime* $\mathfrak{P}$ *of* $k = K(\mu(z_0))$, $\mathfrak{P} \mid \mathfrak{p}$ *of* K, $\mathfrak{p} \mid 3$, *is given by* $\psi_\mathfrak{p}$ *in Table* II.

Finally, we turn from the local exponents to the conductor itself. The conductor is not so easy to find, because the splitting of $\mathfrak{p}$ in k/K is controlled by the order of $\mathfrak{p}$ in the ideal class group; however, its norm can be given easily. The contribution to the norm from primes over $\mathfrak{p}$ is

$$(N\mathfrak{p})^{\psi_\mathfrak{p} \cdot f_\mathfrak{p} \cdot g_\mathfrak{p}} = (N\mathfrak{p})^{\psi_\mathfrak{p} \cdot [k : K(j)][K(j) : K]/e_\mathfrak{p}},$$

TABLE I

Data for Legendre and Jacobi Curves

Case	$[K(\lambda):K(j)]$	$e_{\mathfrak{p}}$	$\psi_{\mathfrak{p}}$: Low	High
1. If $K=\mathbb{Q}(i)$				
and $c=c(2)$	$1//2$	$c(2)$	3	$4\cdot c(2)$
and $c\neq c(2)$	2	$2\cdot c(2)$	4	$8\cdot c(2)-2$
2. If $d\equiv 1, 2, 5, 6 \bmod 8$	2	$2\cdot c(2)$	4	$8\cdot c(2)-2$
$(d\neq 1)$				
3. If $K=\mathbb{Q}(\sqrt{-3})$				
and $c=c(2)$	$1//2$	$c(2)$	2	$2\cdot c(2)$
and $c\neq c(2)$	$3//2$	$3\cdot c(2)$	4	$6\cdot c(2)-2$
4. If $d\equiv 3 \bmod 8$	$3//2$	$3\cdot c(2)$	4	$6\cdot c(2)-2$
$(d\neq 3)$				
5. If $d\equiv 7 \bmod 8$[a]	$1//2$	$c(2)$	0	$2\cdot c(2)$

Note. $a//b$ means "a if $c_2=0$, b if $c_2>0$"; $c(2)=2^{c_2}$ is the power of 2 dividing c.

[a] When $c_2=0$, writing $(2)=\mathfrak{p}_i\mathfrak{p}_j$, then the prime $\mathfrak{p}_i$ for which $\operatorname{ord}_{\mathfrak{p}_i}(z_0)>\operatorname{ord}_{\mathfrak{p}_j}(z_0)$ behaves as though it belongs to the "high" class, and the other prime $\mathfrak{p}_j$ behaves as though it is in the "low" class; the grössencharacter is ramified only above $\mathfrak{p}_i$.

TABLE II

Data for Hesse Curves

Case	$[K(\mu):K(j)]$	$e_{\mathfrak{p}}$	$\psi_{\mathfrak{p}}$
1. If $K=\mathbb{Q}(\sqrt{-3})$			
and $c=c(3)$	$1//3$	$c(3)$	2
and $c\neq c(3)$	3	$3\cdot c(3)$	1
2. If $d\equiv 0 \bmod 3$	3	$3\cdot c(3)$	1
$(d\neq 3)$			
3. If $K=\mathbb{Q}(i)$			
and $c=c(3)$	$2//3$	$2\cdot c(3)$	1
and $c\neq c(3)$	$4//3$	$4\cdot c(3)$	1
4. If $d\equiv 1 \bmod 3$	$4//3$	$4\cdot c(3)$	1
$(d\neq 1)$			
5. If $d\equiv 2 \bmod 3$			
if $c_3=0$, and $(3)=\mathfrak{p}_i\mathfrak{p}_j$ with	1	1	1
$\operatorname{ord}_{\mathfrak{p}_i}(z_0)>0$, $\operatorname{ord}_{\mathfrak{p}_j}(z_0)<0$[a]			
otherwise	$2//3$	$2\cdot c(3)$	0

Note. $a//b$ means "a if $c_3=0$, b if $c_3>0$"; $c(3)=3^{c_3}$ means the power of 3 dividing c.

[a] In the special case, the grössencharacter is ramified only above the prime $\mathfrak{p}_i$, where $\operatorname{ord}_{\mathfrak{p}_i}(z_0)>0$.

where $N\mathfrak{p}$ is the absolute norm of $\mathfrak{p}$; $e_\mathfrak{p}$, $f_\mathfrak{p}$, and $g_\mathfrak{p}$ are the indices of ramification, inertia, and splitting of $\mathfrak{p}$ in $\mathscr{k}/K$, and $j=j(z_0)$ is the j-invariant of the curve. $\psi_\mathfrak{p}$, $[\mathscr{k}: K(j)]$, and $e_\mathfrak{p}$ are given in Tables I and II. ($[\mathscr{k}: K(j)]$ and $e_\mathfrak{p}$ were computed as follows: since $K(j(z_0))$ is the classfield to $K^\times \cdot q_{z_0}^{-1}(\mathbb{U})$, $[\mathscr{k}: K(j)] = [K^\times \cdot q_{z_0}^{-1}(\mathbb{U}): K^\times \cdot q_{z_0}^{-1}(\mathbb{U}_F)]$, while $e_\mathfrak{p}$ is the terminal slope of the ψ-function.) On the other hand, $[K(j): K]$ is known classically to be

$$h \cdot c \cdot [\mathscr{O}_K^\times : \mathscr{O}^\times]^{-1} \cdot \prod_{p|c} \left[1 - \left(\frac{K}{p}\right) p^{-1}\right],$$

where, as above, c is the conductor of the order $\mathscr{O}$ corresponding to the lattice $[z_0, 1]$ in K; h is the class number of K, and $(K/p) = 1$, -1, or 0 accordingly as p splits, is inert, or ramifies in K (cf. [8, pp. 106, 123]).

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